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Evaluation of generalized exponential integrals using multinomial expansion theorems

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An efficient and reliable method is presented for calculations the generalized exponential (GE) integrals. The basic series expressions of the generalized GE integrals are established. Evaluation of GE integrals for different values of the parameters, show the efficiencies of the new approach. The numerical results illustrate clearly a further reduction in calculation times. The relationships obtained are valid for the arbitrary values parameters and the computation results obtained are in good agreement with the literature. Numerical results obtained and comparisons with numerical results from the literature are listed.

KEY WORDS: generalized exponential integrals, elliptic curves, radiative transfer

AMS subject classification: 81-V55, 81V45

1. Introduction

It is well known that in any various fields of theoretical physics, quantum chemistry, theory of transport processes, theory fluid flow and astrophysics the major task involves the efficient and accurate computation of GE integrals [1–12]. As complicated system computation of these integrals becomes one of the most laborious and consuming steps in practical reasons. Especially, GE integrals appear in the evaluation of the derivatives of the L-series of an elliptic curve, in a theory of multiple light scattering, and in radiative transfer problems from astrophysics [12–21].

Significant progress in theoretical methods and computer technology during in past decade allows us to obtain reliable theoretical predictions for molecular systems of unprecedented size. The same factors also changed the area of theoretical physics concerned with prediction of accurate molecular properties. Unfortunately, they also were not entirely successful. To our knowledge, many authors [22–35] have addressed this problem although many improvements have

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been made in the past few years by the use of computer, an efficient general formulae for the calculation of GE integrals is not yet available.

We have had considerable success in using the multinomial expansion theorems in evaluation GE integrals. In order to avoid factorials the formulas have been expressed in terms of binomial coefficients. For quick calculations, the binomial coefficients are stored in the memory of the computer.

The aim of this work is to present a new expansion formula for GE integrals. I am believe that, the derived formulas would be useful for experimental and theory.

2. Expression for generalized exponential integral in terms of multinomial coefficients

The GE integrals are defined as

$$G_k(x) = \frac{1}{(k-1)!}\int_1^\infty \mathrm{e}^{-xy}(\ln y)^{k-1}\frac{\mathrm{d}y}{y}, \tag{1}$$

where $k = 0, 1, \ldots$. Milgram [28] introduced another generalization of the GE integral in the form:

$$E_s^j(x) = \frac{1}{\Gamma(j+1)}\int_1^\infty \mathrm{e}^{-xy}(\ln y)^j\frac{\mathrm{d}y}{y^s}. \tag{2}$$

Consequently, by considering the particular case of equation (2) with $j = k-1$ and $s = 1$, one can obtain the GE integral that is $G_k(x) = E_1^{k-1}(x)$. We notice that, taking into account the formula (2) for $j = 0$ we obtain for well known exponential integral $E_s(x)$:

$$E_s(x) = \int_1^\infty \mathrm{e}^{-xy}\frac{\mathrm{d}y}{y^s}. \tag{3}$$

This integral is the leading term in a transport and fluid problems, especially in astrophysics. In order to established expressions for the GE integrals we shall first considering well known the multinomial expansion theorems and series relationship of the $\ln x$, respectively [36]:

$$\begin{aligned}&(x_1+x_2+x_3+\cdots+x_t)^n\\ &\quad=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\sum_{n_3=0}^{n-n_1-n_2}\cdots\sum_{n_t=n-n_1-n_2-\ldots-n_t}^{n-n_1-n_2-\ldots-n_t} F_{n_1,n_2,n_3,\ldots n_t}(n)x_1^{n_1}x_2^{n_2}x_3^{n_3}\ldots\ldots x_t^{n_t},\end{aligned} \tag{4}$$

and

$$\ln x = \lim_{t\to\infty}\sum_{i=1}^{t}\frac{1}{i}\left(\frac{x-1}{x}\right)^i \quad \text{for} \quad x \geqslant \frac{1}{2}. \tag{5}$$

Here t is upper limit of summation and $F_{n_1,n_2,n_3,\ldots n_t}(n)$ are multinomial coefficients defined by

$$F_{n_1,n_2,n_3,\ldots n_t}(n) = \frac{n!}{(n_1)!(n_2)!(n_3)!\ldots..(n_t)!}. \tag{6}$$

In order to avoid factorials, we can be the expressed the multinomial coefficients (equation (6)) in terms of binomial coefficients:

$$\begin{aligned} &F_{n_1,n_2,n_3,\ldots n_t}(n) \\ &\quad = F_{n_1}(n)F_{n_2}(n-n_1)F_{n_3}(n-n_1-n_2)\ldots..F_{n_t}(n-n_1-n_2-\cdots-n_{t-1}), \end{aligned} \tag{7}$$

where $F_m(n) = n!/[m!(n-m)!]$ are binomial coefficients. For quick calculations, the binomial coefficients are stored in the memory of the computer. For the binomial coefficients we use the following recurrence relation:

$$F_m(n) = F_m(n-1) + F_{m-1}(n-1). \tag{8}$$

In order to put these coefficients into or to get them back from the memory, the positions of certain coefficients $F_m(n)$ are determined by the following relation:

$$F(n,m) = n(n+1)/2 + m + 1. \tag{9}$$

Now we move on the determination of expression for the GE integral, (equation (2)), in terms of exponential integral and multinomial coefficients. Taking into account equations (4) and (5) in equation (2) we obtain for the GE integral the series expansion formulas in terms of multinomial coefficients:

$$\begin{aligned} E_s^n(x) = {} & \frac{1}{\Gamma(n+1)} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \sum_{n_3=0}^{n-n_1-n_2} \cdots \sum_{n_t=n-n_1-n_2-\cdots-n_t}^{n-n_1-n_2-,\ldots,-n_t} \\ & \times \sum_{i=0}^{n_1+2n_2+,\cdots,+tn_t} (-1)^i F_{n_1,n_2,n_3,\ldots,n_t}(n) F_i(n_1+2n_2+\cdots+tn_t) \\ & \times \frac{1}{2^{n_2}3^{n_3}\ldots t^{n_t}} E_{i+s}(x) \end{aligned} \tag{10}$$

where $t = 1, 2, 3, \ldots$ and $E_s(x)$ is the well known exponential integral defined by equation (3). We note that the choice of reliable formulas for evaluation of these auxiliary functions is the prime importance in accurate GS integral calculations. Several procedures for evaluating the exponential integral can be found in the literature [36–41]. In order to evaluate the exponential integral function $E_n(x)$, for $x < 1$ are calculated by the recursive formula [31]:

$$(n-1)E_n(x) = \mathrm{e}^{-x} - xE_{n-1}(x) \tag{11}$$

where the starting term is given by

$$E_1(x) = \int_1^\infty t^{-1} e^{-xt} dt \tag{12}$$

The exponential integral function $E_1(x)$, is evaluated by means of the Gaussian integration described in Ref. [30]. For $x > 1$ we employ the following continued fraction representation [30]:

$$E_n(x) = e^{-x} \left\{ \frac{1}{x+n-} \frac{1 \times n}{x+n+2-} \frac{2(n+1)}{x+n+4-} \cdots \right\}. \tag{13}$$

3. Numerical calculations and discussion

In this section, we describe methods to GE integrals using the formulas presented in the previous section. The computations programs were performed in the Turbo Pascal language and were implemented using a personal computer PENTIUM III 800 MHz. The computations were performed for wide range of GE integral parameters. The algorithm proposed in this work can be useful for the calculation of GE integrals.

Basic formulas that can be efficiently implemented for computer evaluation were obtained for every type of GE integrals. We compared the numerical results obtained using by [33]. As will be clear from our tests that for value $t = 15$ in equation (10), this formula yields significantly accuracy for arbitrary values of integral parameters. Greater accuracy is easily attainable by the use of more terms of expansion (10). Table 1 contains values obtained for the complete expressions of the GE integrals. As can be seen from Table, the calculation results of GE integrals show good rate of convergence with literature under range of parameters. Using the method discussed in this paper, such calculations are not much more difficult. Hopefully, this simplication will persist at higher orders.

Table 1
The values of $E_s^j(x)$ integrals obtained from equation (10) and Ref. [33].

s	j	x	Equation (8)	Ref. [33]
1	3	2	7.73625447084289E–04	0.0007740512
1	3	1	1.10207420567491E–02	0.0110708954
1	3	5	2.90271482349927E–06	0.0000029028
1	3	10	2.09313486108000E–09	0.0000000021
3	2	4.5	2.37968837085995E–05	
5	3	7.4	2.18895135850061E–08	
10	3	3.8	6.58601779248419E–07	
12	2	12.3	3.15418329213879E–10	

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